

A

Dirac: Intro/refresh26/2/21
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Free-Ried Dirac equation

$$(\gamma^\mu p_\mu - m)\psi = 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$(i\cancel{\partial} - m)\psi = 0$$

$$(\gamma^\mu p_\mu + m)(\gamma^\nu p_\nu - m)\psi = 0$$

$$= (\epsilon^2 - \vec{p}^2 - m^2)\psi = 0 \quad ? \quad (\text{K-G equation})$$

$$\Rightarrow (\gamma^0)^2 = 1, \quad (\gamma^a)^2 = -1 \quad a=1,2,3$$

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \quad \text{for } \mu \neq \nu$$

$$\Rightarrow \boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}} \quad (g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \text{ e.g.})$$

Simplest: 4x4

$$\text{Chiral (Weyl) rep: } \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}$$

- Good for high energy / QED

$$\text{Dirac rep: } \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}$$

- Good for atomic: nice non-rel. limit

$$\sigma^a \text{ Pauli spin, e.g. } \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{Also } \sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \quad (\gamma^5)^2 = 1, \quad \{\gamma^5, \gamma^\mu\} = 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Dirac
"bi"-Spinor

$\psi^\dagger \psi$ - not Lorentz Invariant

$$\psi^\dagger \gamma^0 \psi \text{ is } \Rightarrow \boxed{\bar{\psi} \equiv \psi^\dagger \gamma^0}$$

Free-field Lagrangian

$$\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \quad \phi = \begin{cases} \psi \\ \bar{\psi} \leftarrow \text{D.E for } \psi \end{cases}$$

$$q = |e| \\ c = \epsilon_0 = \hbar = 1$$

$\psi \rightarrow \psi e^{-iq\theta(x)}$: must introduce A to make \mathcal{L} invar.

$$\mathcal{L} = i \bar{\psi} \gamma^\mu (\partial_\mu + iqA_\mu) \psi - m \bar{\psi} \psi$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \theta \quad \left(\begin{array}{l} \text{cancels left-over derivative} \\ \text{from } \partial_\mu (\psi e^{-iq\theta}) \end{array} \right)$$

only anti-symmetric combos of ∂A leave \mathcal{L} invariant. factor: by convention

$$\mathcal{L}_D = i \bar{\psi} \gamma^\mu (\partial_\mu + iqA_\mu) \psi - m \bar{\psi} \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\boxed{\mathcal{L}_D = i \bar{\psi} (\not{\partial} - m) \psi - \frac{1}{4} F^2}$$

$$F? \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \quad (1)$$

↳ just from form of anti-symm F

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -q \bar{\Psi} \gamma^\mu \Psi$$

$$\partial_\mu F^{\mu\nu} = \underbrace{q \bar{\Psi} \gamma^\mu \Psi}_{J^\mu} \quad (2)$$

eg. $v=0$ in (2) $\nabla \cdot \vec{E} = \underbrace{\int_S}_{\rho}$

$$\text{if } \left(\begin{array}{l} A_\mu^\mu = (A_0, \vec{A}) \\ \vec{E} \equiv -\nabla A_0 - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} \equiv \nabla \times \vec{A} \end{array} \right)$$

$$\{\lambda, \mu, \nu\} \text{ in } (1) \Rightarrow \{0, 1, 2\} \rightarrow \nabla \times E = -\frac{\partial B}{\partial t}$$

(1) + (2) \Rightarrow Maxwells Eqs.

Free - Dirac Solutions

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

Rest-frame = $p = (m, \vec{0})$

$$(i\gamma^0 \partial_0 - m)\psi = 0$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Block γ -matrix

$$i\partial_0 \psi_L - m\psi_R = 0$$

$$i\partial_0 \psi_R - m\psi_L = 0$$

4 indep solutions: u^\pm, v^\pm

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} e^{imt}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{imt}$$

$u^+ \quad \quad u^- \quad \quad v^+ \quad \quad v^-$

spin $\times 2$, e^-/e^+

$$(i\vec{\gamma} \cdot \nabla + m)\psi = -i\gamma^0 \partial_t \psi \quad (\gamma^0)^2 = 1$$

$$\underbrace{(i\gamma^0 \vec{\gamma} \cdot \nabla + \gamma^0 m)}_{H_D} \psi = -i\partial_t \psi$$

$$H_D u = m u$$

\rightarrow +ve energy electrons

$$H_D v = -m v$$

\rightarrow "-ve energy" positrons

} "-ve energy" \rightarrow opposite charge
+ve energy when QED.

Plane-Wave: Boost

26/2/21

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e.g.
$$U^+(p, x) = C_{\text{Norm}} \begin{pmatrix} 0 \\ (E_p + p_z)/m \\ 0 \end{pmatrix} e^{-ip \cdot x}$$

for $\vec{p} = p_z \hat{z}$

$$U^s(p, x) = U^s(p) e^{-ip \cdot x}$$

Normalisation

$$\bar{u}^r(p) u^s(p) = 2m \delta_{rs}$$

$$\bar{v}^r(p) v^s(p) = -2m \delta_{rs}$$

$$\bar{u} \cdot v = 0 \quad \text{etc.}$$

Complete (in s-space) as:

$$\sum_{s=\pm} u^s(p) \bar{u}^s(p) = \not{\gamma} \cdot p + m$$

$$\sum_{s=\pm} v^s(p) \bar{v}^s(p) = \not{\gamma} \cdot p - m$$

(not proved here)

$$u^s(p) = \begin{pmatrix} \sqrt{E - \sigma \cdot \vec{p}} \chi^s \\ \sqrt{E + \sigma \cdot \vec{p}} \chi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \chi^s \\ -\chi^s \end{pmatrix}$$

$$\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ [e.g.]}$$

↳ eigenstates of σ_z